A note on the well posed anisotropic discrete BVP's

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Abstract

Using the direct method of the calculus of variations we investigate the existence, uniqueness and continuous dependence on parameters for solutions of second order discrete anisotropic equations with Dirichlet boundary conditions.

1 Introduction

Since difference equations serve as mathematical models in diverse areas, such as economy, biology, physics, mechanics, computer science, finance - see [2]- it is of interest to know the conditions which guarantee the existence and uniqueness of solutions and their dependence on parameters. Problems satisfying all three conditions are called well-posed. We consider an anisotropic difference equation

$$\begin{cases} -\Delta\left(h\left(k-1\right)\left|\Delta x(k-1)\right|^{p(k-1)-2}\Delta x(k-1)\right)=\lambda f(k,x(k),u\left(k\right)), \\ x(0)=x(T+1)=0, \end{cases} \tag{1}$$

where $\lambda>0$ is a numerical parameter, $f:Z[0,T+1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $Z[a,b]$ for $a<b$, a,b integers, is a discrete interval $\{a,a+1,...,b\}$; $\Delta x(k-1)=x\left(k\right)-x(k-1)$ is the forward difference operator; p, $h:Z\left[0,T+1\right]\to\mathbb{R}_{+}$, $p^{-}=\min_{k\in Z[0,T+1]}p\left(k\right)>1$; $h^{-}>0$; $u:Z\left[1,T\right]\to\mathbb{R}$ is a parameter. A solution to (1) is a function $x:Z[0,T+1]\to\mathbb{R}$ which satisfies the given equation and the associated boundary conditions. Solutions are investigated

in a space $H = \{x : Z[0, T+1] \to \mathbb{R} : x(0) = x(T+1) = 0\}$ which is a Hilbert space when considered with a norm $||x|| = \left(\sum_{k=1}^{T+1} |\Delta x(k-1)|^2\right)^{1/2}$. With fixed function u and $\lambda > 0$ the functional corresponding to (1) is

$$J_u(x) = \sum_{k=1}^{T+1} \frac{h\,(k-1)}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^{T} F(k, x\,(k)\,, u(k)),$$

where $F(k,x,u) = \int_0^x f(k,t,u)dt$. $J_u : H \to \mathbb{R}$ is continuously Gâteaux differentiable and equating its Gâteaux derivative $J_u^{'}$ to zero

$$\left\langle J_u'(x), y\right\rangle = \sum_{k=1}^{T+1} h\,(k-1)\,|\Delta x(k-1)|^{p(k-1)-2} \Delta x(k-1) \Delta y(k-1)$$

$$-\lambda \sum_{k=1}^{T} f(k, x(k), u\,(k)) y\,(k) = 0 \text{ for } y \in H$$

provides a weak solutions to (1). Summing by parts we see that a weak solution is a strong one- compare with [1], [6], where the weak solutions are obtained. The uniqueness of a solution is implied by the uniqueness of a critical point and this is turn is guaranteed by strict convexity. The assumptions leading to the existence and uniqueness suffice to prove the continuous dependence on parameters.

Continuous versions of (1) are known to be mathematical models of various phenomena arising in the study of elastic mechanics (see [9]) and electrorheological fluids (see [8]). While the research of continuos anisotropic problems has been very abundant (see [4]), the investigations within their discrete counterparts have only begun [1], [6], [7] - where some tools from the critical point theory are applied in order to get the existence and multiplicity of solutions. None of theses sources considers the well-posed problems. In [1] some uniqueness results are given, but these are provided for some special type of nonlinear terms and for weak solutions. We have already undertaken investigations concerning dependence on parameters for discrete problems in [3]. Some uniqueness results for discrete problems can be found in [10], where different approach is applied.

2 Auxiliary results

For the following estimations see [7]. There exist $C_1, C_2 > 0$ such that,

$$\sum_{k=1}^{T+1} |\Delta x(k-1)|^{p(k-1)} \geq C_1||x||^{p^-} - C_2 \text{ for every } x \in H \text{ with } ||x|| \geq 1. \quad (2)$$

There exists $c_m > 0$ such that for any $x \in H$ and any $m \geq 2$

$$\sum_{k=1}^{T} |x(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta x(k-1)|^m \text{ and} \quad (3)$$

$$(T+1)^{\frac{2-m}{2m}} ||x|| \leq \left(\sum_{k=1}^{T+1} |\Delta x(k-1)|^m \right)^{1/m} \leq (T+1)^{\frac{1}{m}} ||x||. \quad (4)$$

Theorem 1 *[5]Let E be a reflexive Banach space. Let $J : E \to \mathbb{R}$, $J \in C^1(E, \mathbb{R})$, be weakly lower semi-continuous, coercive and strictly convex. Then there exists a unique point $x_0 \in E$ such that $\inf_{x \in E} J(x) = J(x_0)$ and $J^{'}(x_0) = 0$.*

3 Main Result

In this note we assume that

H1 *there exist functions $a : Z[1,T] \to \mathbb{R}_+$, $b : Z[1,T] \to \mathbb{R}$, $q : Z[1,T] \to (1, +\infty)$ such that*

$$|f(k,x,u)| \leq a(k)|x|^{q(k)} + b(k) \text{ for all } x, u \in \mathbb{R} \text{ and all } k \in [1,T];$$

H2 *for any fixed $k \in Z[1,T]$ function $x \to f(k,x,u)$ is nonincreasing for all $u \in \mathbb{R}$.*

H3 *$f(k,0,u) \neq 0$ for at least one $k \in Z[1,T]$ and for all $u \in \mathbb{R}$.*

Theorem 2 *Assume that conditions* ***H1-H3*** *hold with either $p^- > q^+ + 1$ and $\lambda > 0$ or $p^- = q^+ + 1$ and $\lambda < \frac{C_1 h^-(q^-+1)}{p^+ a^+ c_{q^+ + 1}}$. Then for each u :*

$Z[1,T] \to \mathbb{R}$ problem (1) has exactly one nontrivial solution x_u. Let $\{u_n\}_{n=1}^{\infty}$ be a convergent sequence of parameters, where $\lim_{n\to\infty} u_n(k) = \overline{u}(k)$ for $k \in Z[1,T]$. For a sequence $\{x_{u_n}\}_{n=1}^{\infty}$ of solutions to problem (1) corresponding to u_n, there exists a convergent subsequence $\{x_{u_{n_i}}\}_{i=1}^{\infty}$ such that its limit $\overline{x}$ solves (1) for $u = \overline{u}$.

Proof. For the existence and uniqueness we apply Theorem 1. Let u be fixed. Assumption **H3** guarantees that all solutions must be nontrivial. $J_u \in C^1(H,\mathbb{R})$ and it is strictly convex since by **H2** the nonlinear terms are convex and since the terms connected with the difference operator are strictly convex. In order to show the coercivity take $||x|| \geq 1$ with $||x||_C = \max_{k\in Z[1,T]} |x(k)| \geq 1$ and observe using (3), (4) that

$$\left|\sum_{k=1}^{T} F(k,x(k),u(k))\right| \leq \sum_{k=1}^{T}\left(\frac{a(k)|x(k)|^{q(k)+1}}{q(k)+1} + b(k)x(k)\right) \leq$$

$$\sum_{k=1}^{T}\left(\frac{a^+}{q^-+1}|x(k)|^{q^++1} + b^+|x(k)|\right) \leq \frac{a^+c_{q^++1}}{q^-+1}(T+1)||x||^{q^++1} + b^+c_1(T+1)||x||.$$

Further by (2) we get

$$J_u(x) \geq \frac{C_1h^-}{p^+}||x||^{p^-} - \lambda\frac{a^+c_{q^++1}}{q^-+1}(T+1)||x||^{q^++1} - \lambda b^+c_1(T+1)||x|| - C_2. \quad (5)$$

Thus $J_u(x) \to \infty$ as $||x|| \to \infty$ in case $p^- > q^+ + 1$ and also $J_u(x) \to \infty$ as $||x|| \to \infty$ in case $p^- = q^+ + 1$ since $\lambda < \frac{C_1h^-(q^-+1)}{p^+a^+c_{q^++1}}$. So problem (1) has exactly one, nontrivial solution $x_u \in H$.

Let $\{x_{u_n}\}_{n=1}^{\infty}$ be the sequence of solutions corresponding to $\{u_n\}_{n=1}^{\infty}$. Suppose that $\{x_{u_n}\}_{n=1}^{\infty}$ is unbounded. Then $||x_{u_n}|| \to \infty$ as $n \to \infty$. Note also that

$$J_u(x_{u_n}) = \inf_{x\in H} J_{u_n}(x) \leq J_{u_n}(0) = 0. \quad (6)$$

Since $\{x_{u_n}\}_{n=1}^{\infty}$ is assumed unbounded there exists N_0 such that $||x_{u_n}|| \geq 1$ and $||x||_C \geq 1$ for $n \geq N_0$. Now by (5) together with (6) we see that for $n \geq N_0$

$$\frac{C_1h^-}{p^+}||x_{u_n}||^{p^-} - \lambda\frac{a^+c_{q^++1}}{q^-+1}(T+1)||x_{u_n}||^{q^++1} - \lambda b^+c_1(T+1)||x_{u_n}|| \leq C_2.$$

Thus $\{x_{u_n}\}_{n=1}^{\infty}$ must be bounded and we reach a contradiction. Hence there exists a constant $\gamma > 0$ such that $||x_{u_n}|| \leq \gamma$ for $n \in \mathbb{N}$ and there exists a

convergent subsequence $\left\{x_{u_{n_i}}\right\}_{i=1}^{\infty}$ whose limit we denote by $\overline{x}$. Let $\{u_{n_i}\}_{i=1}^{\infty}$ be the corresponding subsequence of parameters which obviously converges to $\overline{u}$. For these subsequences we have for $k \in Z[1,T]$

$$-\Delta\left(h\left(k\right)\left|\Delta x_{u_{n_i}}(k-1)\right|^{p(k-1)-2}\Delta x_{u_{n_i}}(k-1)\right)=f\left(k,x_{u_{n_i}}(k),u_{n_i}(k)\right),$$

$$x_{u_{n_i}}(0)=x_{u_{n_i}}(T+1)=0.$$

Taking limits to both sides of the above relation, we see by continuity that (1) holds with $x=\overline{x}$ and $u=\overline{u}$. ■

4 Conclusions

Other variational discrete boundary value problems can be tackled by our approach provided the assumptions imposed allow for the direct method to be applied. We also can double easily our results by using strict concavity and anti-corecivity of a functional J_u^1

$$J_u^1(x)=\lambda\sum_{k=1}^{T}F(k,x\left(k\right),u(k))-\sum_{k=1}^{T+1}\frac{h\left(k\right)}{p(k-1)}|\Delta x(k-1)|^{p(k-1)}.$$

Theorem 2 remains valid with **H1, H3** retained, with **H2** replaced by

H4 *for all* $k\in Z\left[1,T\right]$, $u\in\mathbb{R}$ *function* $x\rightarrow f\left(k,x,u\right)$ *is nonincreasing*

and with the following assumptions upon p and q

$$q^-+1>p^+,\ \lambda>0 \text{ or } q^-+1=p^+,\ \lambda>(T+1)^{\frac{1-p^-}{2p^-}}\frac{h^-\left(q^-+1\right)}{p^+a^+c_{q^++1}}.$$

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